

Error correction for continuous quantum variables

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We propose an error correction coding algorithm for continuous quantum variables. We use this algorithm to construct a highly efficient 5-wavepacket code which can correct arbitrary single wavepacket errors. We show that this class of continuous variable codes is robust against imprecision in the error syndromes. A potential implementation of the scheme is presented.

Quantum computers hold the promise for efficiently factoring large integers [1]. However, to do this beyond a most modest scale they will require quantum error correction [2]. The theory of quantum error correction is already well studied in two-level or spin- $\frac{1}{2}$ systems (in terms of qubits or quantum bits) [2,3,4,5,6,7]. Some of these results have been generalized to higher-spin systems [8,9,10,11]. This work applies to discrete systems like the hyperfine levels in ions but is not suitable for systems with continuous spectra, such as unbound wavepackets. Simultaneously with this paper, Lloyd and Slotine present the first treatment of a quantum error correction code for continuous quantum variables [12], demonstrating a 9-wavepacket code in analogy with Shor's 9-qubit coding scheme [2]. Such codes hold exciting prospects for the *complete* manipulation of quantum systems, including both discrete and continuous degrees-of-freedom, in the presence of inevitable noise [13].

In this letter we consider a highly efficient and compact error correction coding algorithm for continuous quantum variables. As an example, we construct a 5-wavepacket code which can correct arbitrary single-wavepacket errors. We show that such continuous variable codes are robust against imprecision in the error syndromes and discuss potential implementation of the scheme. This paper is restricted to 1-dimensional wavepackets which might represent the wave function of a non-relativistic 1-dimensional particle or the state of a single polarization of a transverse mode of electromagnetic radiation. We shall henceforth refer to such descriptions by the generic term wavepackets [14].

Rather than starting from scratch we shall use some of the theory that has already been given for error correction on qubits. In particular, Steane has noted that the Hadamard transform

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (1)$$

maps phase-flips into bit-flips and can therefore be used to form a class of quantum error correction codes that consist of a pair of classical codes, one for each type of 'flip' [3]. This mapping between phase and amplitude bases is achieved with a rotation about the y -axis by $\pi/2$ radians in the Bloch sphere representation of the state. In analogy, the position and momentum bases of a continuous quantum state may be transformed into each

other by $\pi/2$ rotations in phase-space. This transition is implemented by substituting the Hadamard rotation in the Bloch sphere by a Fourier transform between position and momentum in phase-space. This suggests that we could develop the analogous quantum error correction codes for continuous systems [15].

We shall find it convenient to use a units-free notation where

$$\begin{aligned} \text{position} &= x \times (\text{scale length}) \\ \text{momentum} &= p / (\text{scale length}), \end{aligned} \quad (2)$$

where x is a scaled length, p is a scaled momentum and we have taken $\hbar = 1$. (We henceforth drop the modifier 'scaled'.) The position basis eigenstates $|x\rangle$ are normalized according to $\langle x'|x\rangle = \delta(x' - x)$ with the momentum basis given by

$$|x\rangle = \frac{1}{\sqrt{\pi}} \int dp e^{-2ixp} |p\rangle. \quad (3)$$

To avoid confusion we shall work in the position basis throughout and so define the Fourier transform as an active operation on a state by

$$\hat{\mathcal{F}}|x\rangle = \frac{1}{\sqrt{\pi}} \int dy e^{2ixy} |y\rangle, \quad (4)$$

where both x and y are variables in the position basis. Note that Eqs. (3) and (4) correspond to a change of representation and a physical change of the state respectively.

In addition to the Fourier transform we shall require an analog to the bit-wise exclusive-OR (XOR) gate for continuous variables. The XOR gate has many interpretations including controlled-NOT gate, addition modulo 2 and parity associated with it. Of these interpretations the natural generalization to continuous variables is addition without a cyclic condition. That is, we take

$$\bigoplus |x, y\rangle = |x, x + y\rangle. \quad (5)$$

By removing the cyclic structure of the XOR gate we have produced a gate which is no longer its own inverse. Thus, in addition to the Fourier transform and this generalized XOR gate we include their inverses to our list of useful gates. This generalized XOR operation performs translations over the entire real line, which are related to the infinite additive group on \mathbb{R} . The characters χ of this group

satisfy the multiplicative property $\chi(x+y) = \chi(x)\chi(y)$ for all $x, y \in \mathbb{R}$ and obey the sum rule

$$\frac{1}{\pi} \int_{-\infty}^{\infty} dx \chi(x) = \delta(x), \quad (6)$$

where $\chi(x) = e^{2ix}$. Interestingly, this sum rule has the same form as that found by Chau in higher-spin codes [10]. Once we have recognized the parallel, it is sufficient to take the code of a spin- $\frac{1}{2}$ system as a basis for our continuous-variable code.

Based on these parallel group properties, we are tempted to speculate a much more general and fundamental relation: We conjecture that n -qubit error correction codes can be paralleled with n -wavepacket codes by replacing the discrete-variable operations (Hadamard transform and XOR gate) by their continuous-variable analogs (Fourier transform, generalized-XOR and their inverses). As a last remark before embarking on the necessary substitutions (in a specific example), we point out that the substitution conjecture is only valid for qubit codes whose circuits involve only these (\hat{H} and XOR) elements. We shall therefore restrict our attention to this class of codes.

An example of a suitable 5-qubit code was given by Laflamme et al. [16]. We show an equivalent circuit in Fig. 1 [17]. As we perform the substitutions, we must determine which qubit-XOR gates to replace with the generalized-XOR and which with its inverse. To resolve this ambiguity, two conditions are imposed. First, we demand that the code retain its properties under the parity operation (on each wavepacket). We conclude that either gate may be chosen for the first operation on initially zero-position eigenstates. Ambiguity remains for the last four XOR substitutions. As a second step, the necessary and sufficient condition for quantum error correction [5,6]:

$$\langle x'_{\text{encode}} | \hat{\mathcal{E}}_{\alpha}^{\dagger} \hat{\mathcal{E}}_{\beta} | x_{\text{encode}} \rangle = \delta(x' - x) \lambda_{\alpha\beta}, \quad \forall \alpha, \beta, \quad (7)$$

must be met. Here $|x_{\text{encode}}\rangle$ encodes a single wavepacket's position eigenstate in a multi-wavepacket state, $\hat{\mathcal{E}}_{\alpha}$ is a possible error state that can be handled by the code and $\lambda_{\alpha\beta}$ is a complex constant independent of the encoded states. [Condition (7) says that correctable errors do not mask the orthogonality of encoded states.]

In the case of a single wavepacket error, for our 5-wavepacket code, it turns out that amongst the conditions of Eq. (7) only $\langle x'_{\text{encode}} | \hat{\mathcal{E}}_{4\alpha}^{\dagger} \hat{\mathcal{E}}_{5\beta} | x_{\text{encode}} \rangle$, having errors on wavepackets 4 and 5, is affected by the ambiguity (see detail below). An explicit calculation of *all* the conditions shows that the circuit of Fig. 2 yields a satisfactory quantum error correction code (as do variations of this circuit due to the extra freedom with respect to the choice of operator acting on wavepackets 1-3). By analogy with the results for higher-spin codes, we know that this code is optimal (though not perfect) and that no

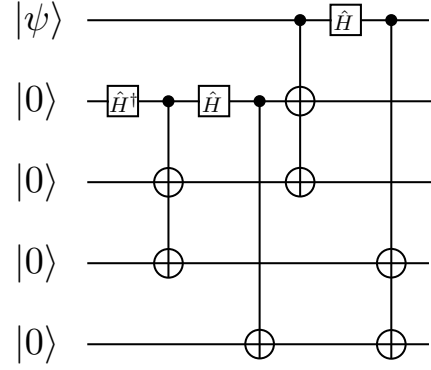


FIG. 1. Quantum error correction circuit from [17]. The qubit $|\psi\rangle$ is rotated into a 5-particle subspace by the unitary operations represented by the operations shown in this circuit. Note that the 3-qubit gates are simply pairs of XORs.

four-wavepacket code would suffice [10]. The code thus constructed has the form

$$|x_{\text{encode}}\rangle = \frac{1}{\pi^{3/2}} \int dw dy dz e^{2i(wy+xz)} \times |z, y+x, w+x, w-z, y-z\rangle. \quad (8)$$

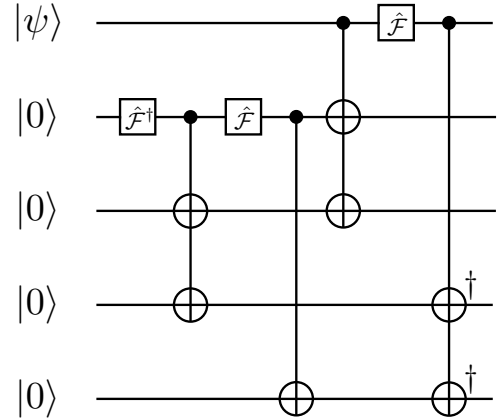


FIG. 2. This ‘circuit’ unitarily maps a 1-dimensional single-wavepacket state $|\psi\rangle$ into a 5-wavepacket error correction code. Here the auxiliary wavepackets $|0\rangle$ are initially zero-position eigenstates. For degrees-of-freedom larger than qubits the ideal XOR is not its own inverse; here the daggers on the XOR gates represent the inverse operation.

Let us demonstrate the calculation of one of the conditions specified by Eq. (7):

$$\begin{aligned} & \langle x'_{\text{encode}} | \hat{\mathcal{E}}_{4\alpha}^{\dagger} \hat{\mathcal{E}}_{5\beta} | x_{\text{encode}} \rangle \\ &= \frac{1}{\pi^3} \int dw' dy' dz' dw dy dz e^{2i(wy+xz-w'y'-x'z')} \\ & \times \delta(z' - z) \delta(y' - y + x' - x) \delta(w' - w + x' - x) \\ & \times \langle w' - z' | \hat{\mathcal{E}}_{\alpha}^{\dagger} | w - z \rangle \langle y' - z' | \hat{\mathcal{E}}_{\beta} | y - z \rangle \end{aligned} \quad (9)$$

$$= \frac{e^{-2i(x'-x)^2}}{\pi^3} \int dw dy dz e^{2i(x'-x)(w+y-z)} \times \langle w - x' + x - z | \hat{\mathcal{E}}_\alpha^\dagger | w - z \rangle \langle y - x' + x - z | \hat{\mathcal{E}}_\beta | y - z \rangle .$$

Making the replacements $w \rightarrow w + z$ and $y \rightarrow y + z$ in this last expression we obtain

$$= \frac{e^{-2i(x'-x)^2}}{\pi^3} \int dw dy dz e^{2i(x'-x)(w+y+z)} \times \langle w - x' + x | \hat{\mathcal{E}}_\alpha^\dagger | w \rangle \langle y - x' + x | \hat{\mathcal{E}}_\beta | y \rangle \quad (10)$$

$$= \frac{\delta(x' - x)}{\pi^2} \int dw dy \langle w | \hat{\mathcal{E}}_\alpha^\dagger | w \rangle \langle y | \hat{\mathcal{E}}_\beta | y \rangle \equiv \delta(x' - x) \lambda_{\alpha\beta} .$$

For the other cases we find by explicit calculation, for wavepackets $j \neq k$, that

$$\langle x'_{\text{encode}} | \hat{\mathcal{E}}_{j\alpha}^\dagger \hat{\mathcal{E}}_{k\beta} | x_{\text{encode}} \rangle = \delta(x' - x) \lambda_{\alpha\beta} . \quad (11)$$

For $j = k$ this constant is found to be

$$\lambda_{\alpha\beta} = \frac{C}{\pi^2} \int dw \langle w | \hat{\mathcal{E}}_\alpha^\dagger \hat{\mathcal{E}}_\beta | w \rangle , \quad (12)$$

where C is formally infinite.

We shall argue that this infinity vanishes when the syndrome is read with only finite precision, which is always going to be the real situation. However, this requires us to demonstrate that our codes are robust: that for a sufficiently good precision we may correct single-wavepacket errors to any specified accuracy. In order to understand how the error syndromes are measured, let us consider a simpler code, namely, the continuous version of Shor's original 9-qubit code:

$$|x_{\text{encode}}\rangle = \frac{1}{\pi^{3/2}} \int dw dy dz e^{2ix(w+y+z)} \times |w, w, y, y, y, z, z, z\rangle , \quad (13)$$

where parity alone removes all ambiguity. (This code has been independently obtained by Lloyd and Slotine [12].) Since this 9-wavepacket code corrects position errors and momentum errors separately, it is sufficient to study the subcode

$$|x_{\text{encode}}\rangle = |x, x, x\rangle , \quad (14)$$

designed to correct position errors on a single wavepacket. The most general position error (on a single wavepacket) is given by some function of the momentum of that system $\hat{\mathcal{E}}(\hat{p})$ and need not be unitary on the code subspace [Eq. (7)]. The action of such an error on a wavepacket may be written in the position basis as

$$\hat{\mathcal{E}}(\hat{p})|x\rangle = \frac{1}{\pi} \int dy dp e^{2ip(y-x)} \mathcal{E}(p)|y\rangle = \int dy \tilde{\mathcal{E}}(y)|x-y\rangle , \quad (15)$$

where $\tilde{\mathcal{E}}(x)$ is the Fourier transform of $\mathcal{E}(p)$. Thus the most general position error looks like a convolution of the wavepacket's ket with some unknown (though not completely arbitrary) function. Suppose this error occurs on wavepacket one in the repetition code (14). Further, let us use auxiliary wavepackets (so-called ancillae) and compute the syndrome as shown in Fig. 3, then the resulting state may be written:

$$\int dy \tilde{\mathcal{E}}(y) |x-y, x, x, -y, 0, y\rangle . \quad (16)$$

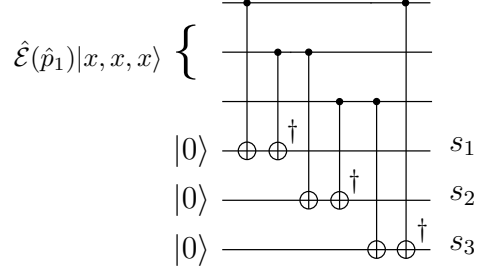


FIG. 3. Syndrome calculation and measurement: A state with a single-wavepacket position error (here on wavepacket 1) enters and the differences of each pair of positions is computed. The syndrome $\{s_1, s_2, s_3\}$ may now be directly measured in the position basis.

Everything up till now has been unitary and assumed ideal. Now measure the syndrome: Ideally it would be $\{-y, 0, y\}$ collapsing the wavepacket for a specific y . Correcting the error is now easy, because we know the location, value and sign of the error. Shifting the first wavepacket by the amount y retrieves the correctly encoded state $|x, x, x\rangle$. Note that this procedure uses only very simple wavepacket-gates: The comparison stage is done *classically*, in contrast to the scheme of Lloyd and Slotine, where the comparison is performed at the amplitude level and involves significantly more complicated interactions [12].

It is now easy to see what imprecise measurements of the syndromes will do. Suppose each measured value of a syndrome s'_j is distributed randomly about the true value s_j according to the distribution $p_{\text{meas}}(s'_j - s_j)$. We find two conditions for error-correction to proceed smoothly. First, $p_{\text{meas}}(x)$ must be narrow compared to any important length scales in $\tilde{\mathcal{E}}(x)$. This guarantees that the chance for ‘correcting’ the wrong wavepacket is negligible and reduces the position-error operator to an uninteresting prefactor. If the original unencoded state had been $\int dx \psi(x)|x\rangle$ then after error correction we would obtain the mixed state

$$\int dx' dx dz \psi(x) \psi^*(x') p_{\text{meas}}(z) \times |x-z, x, x\rangle \langle x'-z, x', x'| . \quad (17)$$

Thus, unless $p_{\text{meas}}(x)$ is *also* narrow compared to any important length scales in $\psi(x)$, decoherence will appear

in the off-diagonal terms for wavepacket 1 of the corrected state (17). This second condition is also seen in the quantum teleportation of continuous variables due to inaccuracies caused by measurement [13]. These conditions roughly match those described by Lloyd and Slotine [12]. We note that any syndrome imprecision will degrade the encoded states, though this precision may be improved by repeated measurements of the syndromes. For our 5-wavepacket example (8), syndromes consist of sums of two or more wavepacket positions or momenta and are measured similarly.

It should be noted that Chau's higher-spin code [10] could have been immediately taken over into a quantum error correction code for continuous quantum variables in accordance with our substitution procedure. However, we have produced an equivalent code with a more efficient circuit prescription: Whereas Chau gives a procedure for constructing his higher-spin code using 9 generalized XOR operations, the circuit in Fig. 2 requires only 7 such gates or their inverses. In fact, we could run this substitution backwards to obtain a cleaner 5-particle higher-spin code based on Eq. (8).

In order to consider potential implementations of the above code let us restrict our attention to a situation where the wavepackets are sitting in background harmonic-oscillator potentials. By the virial theorem the form of a wavepacket in such a potential is preserved up to a trivial rotation in phase-space with time. The two operations required may be implemented simply as follows: The rotation in phase-space, Eq. (4), may be obtained by delaying the phase of one wavepacket relative to the others, and the XOR operation, Eq. (5), should be implemented via a quantum non-demolition (QND) coupling. There exists extensive experimental literature on these operations both for optical fields and for trapped ions [13,18,19,20,21].

The conjecture put forth in this letter leads to a simple, 2-step design of error correction codes for continuous quantum variables. According to this conjecture, any qubit code, whose circuit operations include only a specific Hadamard transformation, its inverse and the ideal XOR, may be translated to a continuous quantum-variable code, by substituting these operators with their continuous analogs and then imposing two criteria – parity invariance and the error-correction condition – which remove any ambiguities in the choice of operators. We demonstrate the success of this coding procedure in two examples (one based on Shor's 9-qubit code [2], and a second based on a variation of the Laflamme et al. 5-qubit code [16,17]). The 5-wavepacket code presented here is the optimal continuous encoding of a single 1-dimensional wavepacket that protects against arbitrary single-wavepacket errors. We show that this code (and in fact the entire class of codes derived in this manner) are robust against imprecision in the error syndromes. The potential implementation of the proposed class of

circuits in optical-field and ion-trap set-ups is an additional incentive for further investigation of the robust manipulation of continuous quantum variables.

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- [1] P. W. Shor in *Proc. 35th Annual Symposium on the Foundations of Computer Science*, edited by S. Goldwasser (IEEE Computer Society Press, Los Alamitos, California, 1994), p.124.
 - [2] P. W. Shor, *Phys. Rev. A* **52**, R2493 (1995).
 - [3] A. M. Steane, *Proc. Roy. Soc. London* **452**, 2551 (1996).
 - [4] A. R. Calderbank and P. W. Shor, *Phys. Rev. A* **54**, 1098 (1996).
 - [5] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
 - [6] E. Knill and R. Laflamme, *Phys. Rev. A*, **55**, 900 (1997).
 - [7] A. R. Calderbank, E. M. Rains, P. W. Shor and N. J. A. Sloane, *Phys. Rev. Lett.* **78**, 405 (1997).
 - [8] E. Knill, LANL report LAUR-96-2717, preprint quant-ph/9608048.
 - [9] H. F. Chau, *Phys. Rev. A* **55**, R839 (1997).
 - [10] H. F. Chau, *Phys. Rev. A* **56**, R1 (1997).
 - [11] E. M. Rains, LANL preprint quant-ph/9703048.
 - [12] S. Lloyd and J.-J. E. Slotine, LANL preprint quant-ph/9711021.
 - [13] S. L. Braunstein and H. J. Kimble, "Teleportation of continuous quantum variables," *Phys. Rev. Lett.*, submitted; S. L. Braunstein, H. J. Kimble, Y. Sorensen, A. Furusawa and N. Ph. Georjades, "Teleportation of continuous quantum variables," IQEC 1998, abstract submitted.
 - [14] Although we consider multi-wavepacket states as one-dimensional we would not want them to physically overlap so they could, for example, be displaced one from another in an orthogonal direction.
 - [15] An example using as few as 7 qubits to correct an arbitrary single-qubit error in a 1-qubit encoded state can be found in [3].
 - [16] R. Laflamme, C. Miquel, J. P. Paz and W. H. Zurek, *Phys. Rev. Lett.* **77**, 198 (1996).
 - [17] S. L. Braunstein and J. A. Smolin, *Phys. Rev. A* **55**, 945 (1997).
 - [18] S. F. Pereira, Z. Y. Ou and H. J. Kimble, *Phys. Rev. Lett.* **72**, 214 (1994).
 - [19] K. Bencheikh, J. A. Levenson, P. Grangier and O. Lopez, *Phys. Rev. Lett.* **75**, 3422 (1995).
 - [20] R. L. de Matos Filho and W. Vogel, *Phys. Rev. Lett.* **76**, 4520 (1996).
 - [21] R. Bruckmeier, H. Hansen and S. Schiller, *Phys. Rev. Lett.* **79**, 1463 (1997).